Propagation of Energy Pulses in Absorbing/Viscous Material Media

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Abstract

A delta-pulse that emerges from the origin of coordinates in a lossy, viscous, homogeneous medium is studied as it propagates through the medium experiencing absorption and dispersion. Elementary models for such propagation problems have appeared in classical textbooks (e.g., J.D. Jackson, *Classical Electrodynamics*, pp. 212-215, Wiley, 1962). Generally speaking, as the pulse propagates, its amplitude decreases, and its width broadens. A more detailed, exact analysis of this problem emerges from the study presented here in which the pulse obeys a third-order partial differential equation of parabolic type in time and one space dimension, first obtained by Stokes in 1885. We obtain the exact solution of the pertinent boundary and initial value problem (BIVP) posed in a rigorous fashion, in which the initial displacement is the delta-pulse in question. The resulting exact distortion of the pulse (i.e., amplitude-decaying and width-broadening shape) emerges from a series solution which we have obtained by Laplace transform techniques. The solution exhibits the expected smoothing-out effects of dispersion. A portion of the final expression, which contains a sum of repeated integrals of the complementary error function, is recast in power series form, thus simplifying the final result. We also show how to obtain an approximate solution of the present BIVP by means of the method of steepest descents. This approximation agrees with the leading term of the exact solution described above. Further examination of the dispersion relation associated with the governing PDE shows that the kinematic viscosity of the medium must be related to the pulse propagation speed in a specific way to insure that propagation with attenuation actually occurs. Under this simple but restrictive condition, quantitative details of the distorted pulse amplitude are illustrated in several nondimensional graphs. These are plotted versus time (or position) for discrete successive values of the spatial position (or time).

1 Introduction

The propagation of a sound pulse through a viscous fluid is an important problem that has been often revisited since the governing equation for the propagation was established by Stokes [1] in 1885. Various, more or less generalized, formulations have appeared in textbooks [2, 3, 4, 5] as well as in papers [6, 7, 8]. The governing equation is a third-order partial differential equation of

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parabolic type. In one space dimension, the boundary/initial value problem is

$$\begin{cases} \dfrac{\nu}{c^2}\dfrac{\partial^3 u}{\partial t\,\partial x^2} + \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} \quad (x > 0,\ t > 0), \\ u(x,0) = 0, \quad \dot{u}(x,0) = 0, \\ u(0,t) = u_0 \delta(t), \end{cases} \tag{1}$$

where $u(x,t)$ is the fluid velocity, c is the sound speed, ν is the kinematic viscosity coefficient responsible for the attenuation, $\dot{u}$ is the acceleration, and $\delta(t)$ is the Dirac delta function. Attempts to solve this equation exactly, in analytic form, have been unsuccessful [7]. Of particular interest is the solution when the boundary condition is an impulsive excitation at the origin. If such solution were obtained, then it could be used, by means of the convolution theorem, to find solutions for any other excitation. This basic solution will reveal the damping effects of viscosity in a quantitative way. Qualitative analyses in the literature [9] have exhibited the anticipated behavior in which the pulse amplitude decreases, and its width broadens, as it propagates.

We note that parabolic PDEs similar to this often have precursor solutions with infinite speed of propagation, so that disturbances are felt immediately at all points in the fluid. This property allows for the smothing-out effect of dispersion within the context of continuum mechanics. This issue has been covered in detail by Weymann in 1967 [10] and will not be considered here. In the following sections, we derive the analytic solution obtained by operational methods as well as an approximation to the analytical solution based on the method of steepest descents, which will identify the strongest contribution within the complete analytic solution to the value of the pertinent contour integral.

2 Theory

A Laplace transform pair is here defined as

$$\begin{cases} \hat{f}(s) = \displaystyle\int_0^\infty f(t) e^{-st}\, dt, \\ f(t) = \dfrac{1}{2\pi i} \displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} \hat{f}(s) e^{st}\, ds, \end{cases} \tag{2}$$

where

$$\mathcal{L}\left\{f'(t)\right\} = s\hat{f}(s) - f(0+). \tag{3}$$

We take the Laplace transform of Eq. 1 with respect to time t to obtain

$$\begin{cases} (1 + \tau s)\hat{u}_{xx}(x,s) - \dfrac{s^2}{c^2}\hat{u}(x,s) = 0, \\ \hat{u}(0,s) = u_0, \end{cases} \tag{4}$$

where $\hat{u}(x,s)$ is the transform of $u(x,t)$, subscripts denote partial differentiation, $\tau = \nu/c^2$, and, because of the initial condition, we take $u_{xx}(x,0) = 0$. Since ν has the units $[L^2/T]$, τ has the units of time. Because of the transform, u_0 now takes the units of length. The solution of this system is

$$\hat{u}(x,s) = u_0 \exp\left(-\frac{sx}{c\sqrt{1 + \tau s}}\right). \tag{5}$$

We now take the inverse Laplace transform of this last equation to obtain

$$u(x,t) = u_0 \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{-\frac{sx}{c\sqrt{1+\tau s}}} e^{st}\, ds, \tag{6}$$

which has an essential singularity at $s = -1/\tau$. With the change of variable

$$s = p - 1/\tau, \tag{7}$$

Eq. 6 becomes

$$u(x,t) = u_0 \frac{e^{-t/\tau}}{2\pi i} \int_{\sigma+1/\tau - i\infty}^{\sigma+1/\tau + i\infty} e^{\frac{x}{c\tau}\left(\frac{1}{\sqrt{\tau p}} - \sqrt{\tau p}\right)} e^{pt}\, dp. \tag{8}$$

We now define the nondimensional variables

$$\bar{x} = \frac{x}{c\tau}, \qquad \bar{t} = \frac{t}{\tau}, \tag{9}$$

in which case Eq. 8 becomes

$$\begin{aligned} u(x,t) &= u_0 \frac{e^{-\bar{t}}}{2\pi i} \int_{\sigma+1/\tau - i\infty}^{\sigma+1/\tau + i\infty} e^{\bar{x}\left(\frac{1}{\sqrt{\tau p}} - \sqrt{\tau p}\right)} e^{pt}\, dp & (10) \\ &= u_0 \frac{e^{-\bar{t}}}{2\pi i} \int_{\sigma+1/\tau - i\infty}^{\sigma+1/\tau + i\infty} F_1(p) F_2(p) e^{pt}\, dp, & (11) \end{aligned}$$

where

$$F_1(p) = e^{-\bar{x}\sqrt{\tau p}}, \qquad F_2(p) = e^{\bar{x}/\sqrt{\tau p}}. \tag{12}$$

If we expand $F_2(p)$ in a Maclaurin series, the product of these two functions can be written

$$\begin{aligned} F(p) &= F_1(p) F_2(p) & (13) \\ &= e^{-\bar{x}\sqrt{\tau p}} \left[1 + \frac{\bar{x}}{\sqrt{\tau}\sqrt{p}} + \frac{\bar{x}^2}{2!\tau p} + \cdots + \frac{\bar{x}^n}{n!\tau^{n/2} p^{n/2}} + \cdots \right] & (14) \\ &= e^{-\bar{x}\sqrt{\tau p}} + \frac{\bar{x}}{\sqrt{\tau}} \cdot \frac{e^{-\bar{x}\sqrt{\tau p}}}{\sqrt{p}} + \frac{\bar{x}^2}{2!\tau} \cdot \frac{e^{-\bar{x}\sqrt{\tau p}}}{p} + \cdots + \frac{\bar{x}^n}{n!\tau^{n/2}} \cdot \frac{e^{-\bar{x}\sqrt{\tau p}}}{p^{n/2}} + \cdots, & (15) \end{aligned}$$

which can be inverted term-by-term.

We note that the first term in Eq. 15 is $F_1(p)$, whose inverse $f_1(t)$ is tabulated [11, 12]; hence,

$$f_1(t) = \frac{\bar{x}}{2\tau\bar{t}\sqrt{\pi\bar{t}}} e^{-\frac{\bar{x}^2}{4\bar{t}}}. \tag{16}$$

The inverse of the second term in Eq. 15 is also tabulated [11]:

$$\mathcal{L}^{-1}\left\{ \frac{\bar{x}}{\sqrt{\tau}} \cdot \frac{e^{-\bar{x}\sqrt{\tau p}}}{\sqrt{p}} \right\} = \frac{\bar{x}}{\tau\sqrt{\pi\bar{t}}} e^{-\bar{x}^2/(4\bar{t})}. \tag{17}$$

Third and subsequent terms in Eq. 15 can be inverted using the known formula [11]

$$\mathcal{L}^{-1}\left\{ \frac{e^{-\beta\sqrt{p}}}{p^{n/2}} \right\} = (4t)^{n/2-1}\, i^{n-2}\, \mathrm{erfc}\left(\frac{\beta}{2\sqrt{t}} \right), \tag{18}$$

where $\beta = \bar{x}\sqrt{\tau}$, and $i^m \operatorname{erfc}(z)$ is a symbolic operator that denotes the mth successive integrals of the complementary error function. We recall that

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt \tag{19}$$

and the recursion relation [13]

$$i^n \operatorname{erfc}(z) = \int_z^\infty i^{n-1} \operatorname{erfc}(t)\, dt, \quad (n = 0, 1, 2, \ldots). \tag{20}$$

Hence the inverse of the nth term in Eq. 15 is, in nondimensional form,

$$\frac{\bar{x}^n}{n!\tau^{n/2}} \mathcal{L}^{-1}\left\{\frac{-e^{-\bar{x}\sqrt{\tau p}}}{p^{n/2}}\right\} = \frac{\bar{x}^n}{\tau n!}(4\bar{t})^{n/2-1}\, i^{n-2} \operatorname{erfc}\left(\frac{\bar{x}}{2\sqrt{\bar{t}}}\right). \tag{21}$$

The complete inversion is thus obtained by combining Eqs. 16 and 17 with the sum of all $n \geq 2$ terms in Eq. 21:

$$u(\bar{x}, \bar{t}) = u_0 \frac{e^{-\bar{t}}}{\tau}\left[\frac{\bar{x}}{\sqrt{\pi\bar{t}}}\left(\frac{1}{2\bar{t}} + 1\right) e^{-\bar{x}^2/(4\bar{t})} + \sum_{n=2}^{\infty} \frac{\bar{x}^n}{n!}(4\bar{t})^{n/2-1}\, i^{n-2} \operatorname{erfc}\left(\frac{\bar{x}}{2\sqrt{\bar{t}}}\right)\right], \tag{22}$$

or, in nondimensional form,

$$\bar{u}(\bar{x}, \bar{t}) = e^{-\bar{t}}\left[\frac{1}{\sqrt{\pi}}(1 + 2\bar{t})\frac{\bar{x}}{2\sqrt{\bar{t}}} e^{-\bar{x}^2/(4\bar{t})} + \sum_{n=2}^{\infty} \frac{2^{n-2}}{\Gamma(n+1)} \bar{x}^n \bar{t}^{n/2}\, i^{n-2} \operatorname{erfc}\left(\frac{\bar{x}}{2\sqrt{\bar{t}}}\right)\right]. \tag{23}$$

The power series expansion of $i^m \operatorname{erfc}(z)$ is given by [11]

$$i^m \operatorname{erfc}(z) = \frac{1}{2^m} \sum_{k=0}^{\infty} \frac{(-2z)^k}{\Gamma(k+1)\Gamma\left(1 + \frac{m-k}{2}\right)}, \tag{24}$$

which can be substituted into Eq. 23 to obtain

$$\bar{u}(\bar{x}, \bar{t}) = e^{-\bar{t}}\left[\frac{1}{\sqrt{\pi}}(1 + 2\bar{t})\frac{\bar{x}}{2\sqrt{\bar{t}}} e^{-\bar{x}^2/(4\bar{t})} + \sum_{n=2}^{\infty} \frac{\bar{x}^n}{\Gamma(n+1)}(\bar{t})^{n/2} \sum_{k=0}^{\infty} \frac{(-1)^k (\bar{x}/\sqrt{\bar{t}})^k}{\Gamma(k+1)\Gamma\left(\frac{n-k}{2}\right)}\right] \tag{25}$$

The triple factorial growth in the denominator of the double sum is an indication of the negligible value of its contribution to the solution. In terms of the original, dimensional variables x and t, this result is

$$\begin{aligned} u(x,t) \;&=\; u_0 e^{-t/\tau}\frac{x}{2c\sqrt{\pi\tau t}}\left(\frac{1}{t} + \frac{2}{\tau}\right) e^{-x^2/(4c^2\tau t)} \\ &\quad + u_0 \frac{e^{-t/\tau}}{t} \sum_{n=2}^{\infty} \frac{1}{\Gamma(n+1)}\left(\frac{t}{\tau}\right)^{n/2}\left(\frac{x}{c\sqrt{\tau t}}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma\left(\frac{n-k}{2}\right)}\left(\frac{x}{c\sqrt{\tau t}}\right)^k. \end{aligned} \tag{26}$$

The factor $e^{-t/\tau}$ comes from the shift in Eq. 7 when changing the variable from s to p. Here it is more convenient to deal with and plot the nondimensional Eq. 25, since a single set of curves will quantitatively describe results for all dimensional values of x and t.

We note that, for $n = m+2$, the series in Eq. 23 can be expressed as

$$\sum_{m=0}^{\infty} (2\bar{x}\bar{t})^m \frac{\bar{x}^2}{\Gamma(m+3)}\, i^m \,\mathrm{erfc}(z),$$

where $z = \bar{x}/(2\sqrt{\bar{t}})$. A convenient way to evaluate the series is by means of the recursion relation [11, 13]

$$i^m \,\mathrm{erfc}(z) = \frac{1}{2m}\, i^{m-2} \,\mathrm{erfc}(z) - \frac{z}{m}\, i^{m-1} \,\mathrm{erfc}(z). \tag{27}$$

All $m \geq 1$ terms can be expressed in terms of the first two by the above relation, where the first two are

$$i^{-1} \,\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\, e^{-z^2}, \quad i^0 \,\mathrm{erfc}(z) = \mathrm{erfc}(z). \tag{28}$$

3 An Approximation

Another way to proceed with the inversion of Eq. 6 is to use the change of variable from s to p given by

$$1 + \tau s = p^2, \tag{29}$$

which yields

$$u(x,t) = u_0 \frac{1}{\pi i \tau} \int_L \exp\left[\frac{1}{\tau}(p^2-1)\left(t - \frac{x}{cp}\right)\right] p\, dp. \tag{30}$$

In this integral, L is a vertical path to the right of the origin O, and there are an essential singularity at $p = 0$ and a branch point at $s = -1/\tau$.

Let $f(p)$ denote the function in the exponent of Eq. 30,

$$f(p) = \frac{1}{\tau}(p^2-1)\left(t - \frac{x}{cp}\right) = \frac{t}{\tau}(p^2-1)\left(1 - \frac{\alpha}{p}\right), \tag{31}$$

where $\alpha = x/(ct)$, from which it follows that

$$\frac{\tau}{t} f'(p) = 2p - \alpha - \frac{\alpha}{p^2}, \tag{32}$$

$$\frac{\tau}{t} f''(p) = 2\left(1 + \frac{\alpha}{p^3}\right). \tag{33}$$

The roots of $f'(p)$ are the solutions of

$$2p^3 - \alpha p^2 - \alpha = 0, \tag{34}$$

which has no negative real roots. We note that

$$f'(1) = 2 - 2\alpha, \quad f'(\alpha) = \alpha(\alpha^2 - 1), \quad f'(\infty) = \infty. \tag{35}$$

There are various cases of interest. Assume $\alpha > 1$, in which case the waveform that started at $t = 0$ has not yet reached the spatial location x, since $x > ct$. In that case, from Eq. 35,

$$f'(1) < 0, \quad f'(\alpha) > 0, \quad f'(\infty) > 0, \tag{36}$$

which, because of the sign change, implies a root between $p = 1$ and $p = \alpha$. We denote that root p_0 (Fig. 1). Similarly, if $\alpha < 1$, there is also a root between $p = 1$ and $p = \alpha$, except that, in

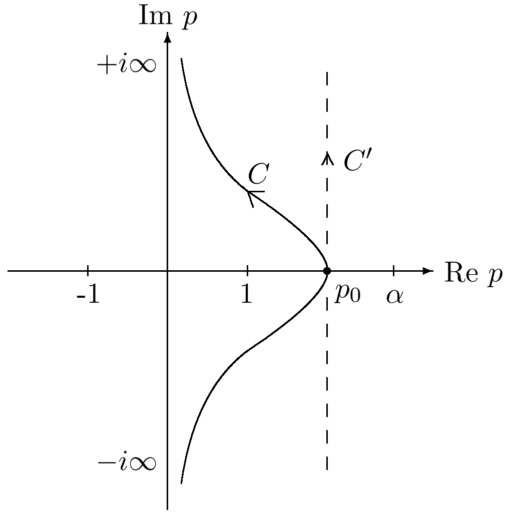

Figure 1: The p-plane with the locations of p_0, α, and the path C of steepest descents.

this case, $p = \alpha$ and $p = 1$ would exchange places in Fig. 1. In both cases, $f''(p_0) > 0$. Thus the path C of steepest descents is parallel to the imaginary axis. This path passes through p_0 and is asymptotic to the imaginary axis for $p \to \pm\infty$. The portion of the path that contributes most to the value of the integral is that near the saddle point p_0. The other roots of $f'(p) = 0$ are complex with negative real parts.

We now write the complex number p in terms of its real and imaginary parts,

$$p = p_0 + i\sigma, \tag{37}$$

so that Eq. 30 can be written

$$u(x,t) = u_0 \frac{1}{\pi i \tau} \int_C \exp\left\{ \frac{t}{\tau}[(p_0 + i\sigma)^2 - 1] \left[1 - \frac{x}{ct(p_0 + i\sigma)} \right] \right\} (p_0 + i\sigma) i \, d\sigma. \tag{38}$$

We can then expand and approximate the exponential to obtain

$$u(x,t) \approx u_0 \frac{p_0}{\pi \tau} \int_{C'} \exp\left[f(p_0) - \frac{t}{\tau}\sigma^2 \left(1 + \frac{\alpha}{p_0^3} \right) \right] d\sigma, \tag{39}$$

where $\alpha = x/(ct)$. Thus,

$$u(x,t) \approx u_0 \frac{p_0 e^{f(p_0)}}{\pi \tau} \int_{-\infty}^{+\infty} e^{-A\sigma^2} d\sigma, \tag{40}$$

where

$$f(p_0) = \frac{t}{\tau}(p_0^2 - 1)\left(1 - \frac{\alpha}{p_0} \right) \tag{41}$$

and p_0 is the root of $f'(p) = 0$ between $p = 1$ and $p = \alpha$. Since the integral in Eq. 40 has the known value $\sqrt{\pi/A}$, we obtain as the final result

$$u(x,t) \approx u_0 \frac{p_0 e^{f(p_0)}}{\sqrt{\pi \tau t}} \sqrt{\frac{p_0^3}{\alpha + p_0^3}} \quad (\alpha > 1). \tag{42}$$

The dependence on x appears only through α, which is required to find p_0. This solution is singular at $t = 0$, which is the time at which the impulse $\delta(t)$ acts.

Eq. 42 is also valid for $\alpha < 1$. In that case, C' does not cross the essential singularity at $p = 0$. However, Eq. 42 is not valid for $\alpha = 1$. For this case, Eq. 34 implies

$$2p^3 - p^2 - 1 = 0, \tag{43}$$

which has one real root $p_0 = 1$ and two complex roots $(-1 \pm i\sqrt{3})/4$. It is convenient for this case to use the changes of variable

$$p = 1 + v, \tag{44}$$

which will have the effect of introducing a shift, and

$$t = t' + \frac{x}{c}. \tag{45}$$

With these changes of variables, Eq. 30 becomes

$$u(x,t) = u_0 \frac{1}{\pi i \tau} \int_L \exp\left[\frac{1}{\tau}(v^2 + 2v)\left(t' + \frac{x}{c} - \frac{x}{c(1+v)}\right)\right](1+v)\,dv. \tag{46}$$

For $v << 1$, this integral simplifies to

$$u(x,t) \approx u_0 \frac{1}{2\pi i \tau} \int_{-i\infty}^{+i\infty} \exp\left[\frac{2v}{\tau}\left(t' + \frac{2xv}{c}\right)\right] dv. \tag{47}$$

With the additional change of variable

$$v = \frac{\sqrt{c\tau p}}{2}, \tag{48}$$

Eq. 47 becomes

$$u(x,t) \approx u_0 \frac{1}{2\pi i} \cdot \frac{\sqrt{c\tau}}{2\tau} \int_{-i\infty}^{+i\infty} \exp\left[xp + t'\sqrt{\frac{cp}{\tau}}\right] \frac{dp}{\sqrt{p}}, \tag{49}$$

which is in the form of an inverse Laplace transform in x [14, 12]. Thus,

$$u(x,t) \approx u_0 \frac{1}{2}\sqrt{\frac{c}{\tau}}\, \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{p}} e^{-k\sqrt{p}}\right\}, \tag{50}$$

where

$$k = -t'\sqrt{\frac{c}{\tau}}. \tag{51}$$

Since this transform is tabulated [11], we obtain

$$u(x,t) \approx u_0 \sqrt{\frac{c}{4\pi\tau x}} \exp\left(-\frac{ct'^2}{4\tau x}\right). \tag{52}$$

However, since $\alpha = 1$ (and $x = ct$), it therefore follows that

$$u(x,t) \approx u_0 \frac{e^{-t/\tau}}{\sqrt{4\pi\tau t}} \exp\left(-\frac{x^2}{4c^2\tau t}\right), \tag{53}$$

where we have included the $e^{-t/\tau}$ factoring resulting from the "shift." This result coincides with the first term of Eq. 26 with $x = ct$ in the amplitude factor. Thus, the first two terms of the earlier solution are the strongest contributors to the inversion integral [15].

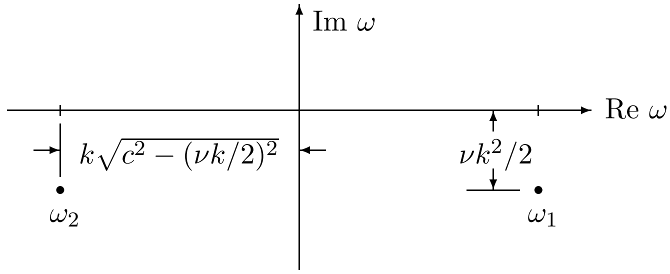

Figure 2: The ω solutions of the dispersion relation.

4 Dispersion Relation

If we seek a solution of Eq. 1a in the form

$$u(x,t) = e^{i(kx-\omega t)}, \tag{54}$$

where $k = \omega/c$, we obtain the quadratic

$$\omega^2 + (i\nu k^2)\omega - k^2c^2 = 0, \tag{55}$$

whose solutions are

$$\omega_{1,2} = \pm k\sqrt{c^2 - (\nu k/2)^2} - i\nu k^2/2. \tag{56}$$

If $c > \nu k/2$, the solutions are complex with a negative imaginary part, as illustrated in Fig. 2. In this case, there is wave propagation through the medium with an attenuation factor $e^{-\nu k^2 t/2}$, which reduces the wave amplitude with time. The second possibility, $c < \nu k/2$, is of no interest, since the ω solutions are purely imaginary, and there is no wave propagation. This situation then resembles ordinary diffusion.

Eq. 55 can also be written in terms of $\tau = \nu/c^2$, in which case

$$\omega^2 = (kc)^2(1 - i\omega\tau), \tag{57}$$

which can be solved for k:

$$k = \frac{\omega}{c\sqrt{1+(\omega\tau)^2}}\sqrt{1+i\omega\tau} \tag{58}$$

$$= \frac{\omega}{c\sqrt{2}\sqrt{1+(\omega\tau)^2}}\left[\sqrt{\sqrt{1+(\omega\tau)^2}+1} + i\sqrt{\sqrt{1+(\omega\tau)^2}-1}\right]. \tag{59}$$

If we now express this complex propagation constant in the usual form

$$k = \frac{\omega}{c_p} + i\alpha, \tag{60}$$

where c_p is the phase velocity, and α is the attenuation, we obtain

$$c_p = \frac{c\sqrt{2}\sqrt{1+(\omega\tau)^2}}{\sqrt{\sqrt{1+(\omega\tau)^2}+1}}, \quad \alpha = \frac{\omega}{c\sqrt{2}} \cdot \frac{\sqrt{\sqrt{1+(\omega\tau)^2}-1}}{\sqrt{1+(\omega\tau)^2}}. \tag{61}$$

We can also write the attenuation factor α in terms of the phase velocity:

$$\alpha = \frac{\sqrt{1+(\omega\tau)^2}-1}{c_p\tau}. \tag{62}$$

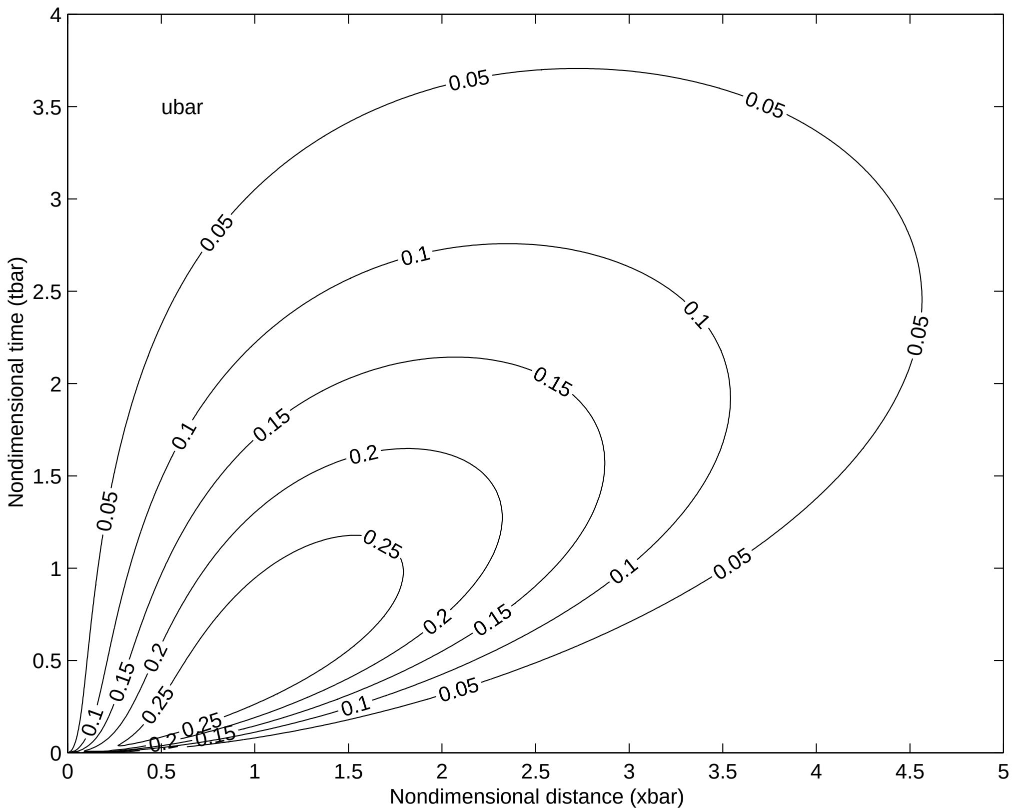

Figure 3: Contour plot of nondimensional velocity vs. nondimensional distance and time.

5 Numerical Results and Discussion

We have obtained both exact and approximate solutions for the propagation behavior of a delta function pulse that travels in a viscous and dispersive medium obeying a general boundary and initial value problem governed by a third-order partial differential equation. The exact solution was obtained by operational methods, where the resulting infinite series was expressed in terms of repeated integrals of the complementary error function. It was also shown how to tranform this series into a rapidly convergent power series.

The approximate solution was found independently by the method of stationary phase. It agrees exactly with the first term of the series solution found before.

We display these results in several plots involving the nondimensional solution presented in Eq. 25. In Fig. 3 is shown a contour plot of the first two terms (omitting the double sum) of the expression for the nondimensional fluid particle velocity $\bar{u}$ (denoted *ubar* in the figure) *vs.* nondimensional distance *xbar* from the origin and nondimensional time *tbar*. In Figs. 4 and 5, we show various cuts taken from Fig. 3. In Fig. 4, nondimensional velocity is plotted *vs.* nondimensional time for several values of nondimensional distance. In Fig. 5, nondimensional velocity is plotted *vs.* nondimensional distance for several values of nondimensional time. All three plots show that, as the initial impulse advances either in time or space, its peak amplitude decays, and its width broadens, as would be expected qualitatively. However, the formulas derived (along with their corresponding plots) represent a *quantitative* evaluation of the anticipated effects of viscosity and dispersion.

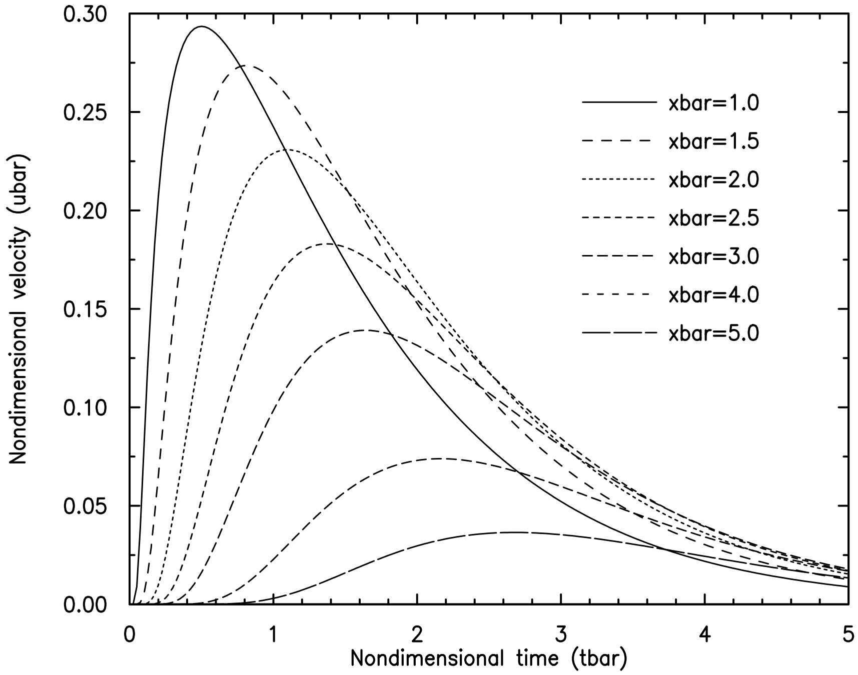

Figure 4: Plot of nondimensional velocity vs. nondimensional time for several values of nondimensional distance.

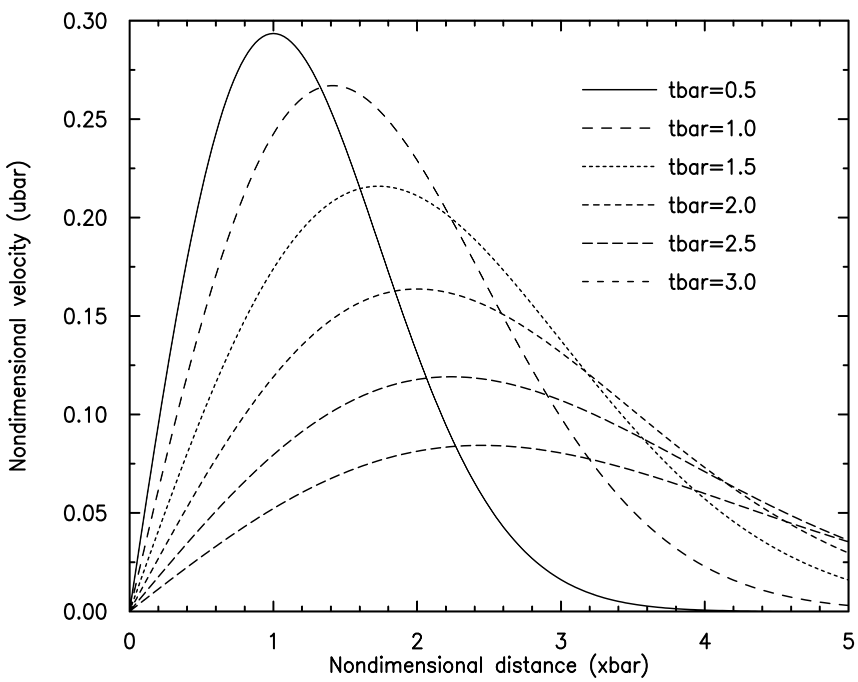

Figure 5: Plot of nondimensional velocity vs. nondimensional distance for several values of nondimensional time.

Thus, we have analytically and quantitatively described the decay and broadening of impulses propagating in viscous media as modeled by Stokes' classical boundary-initial value problem.

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